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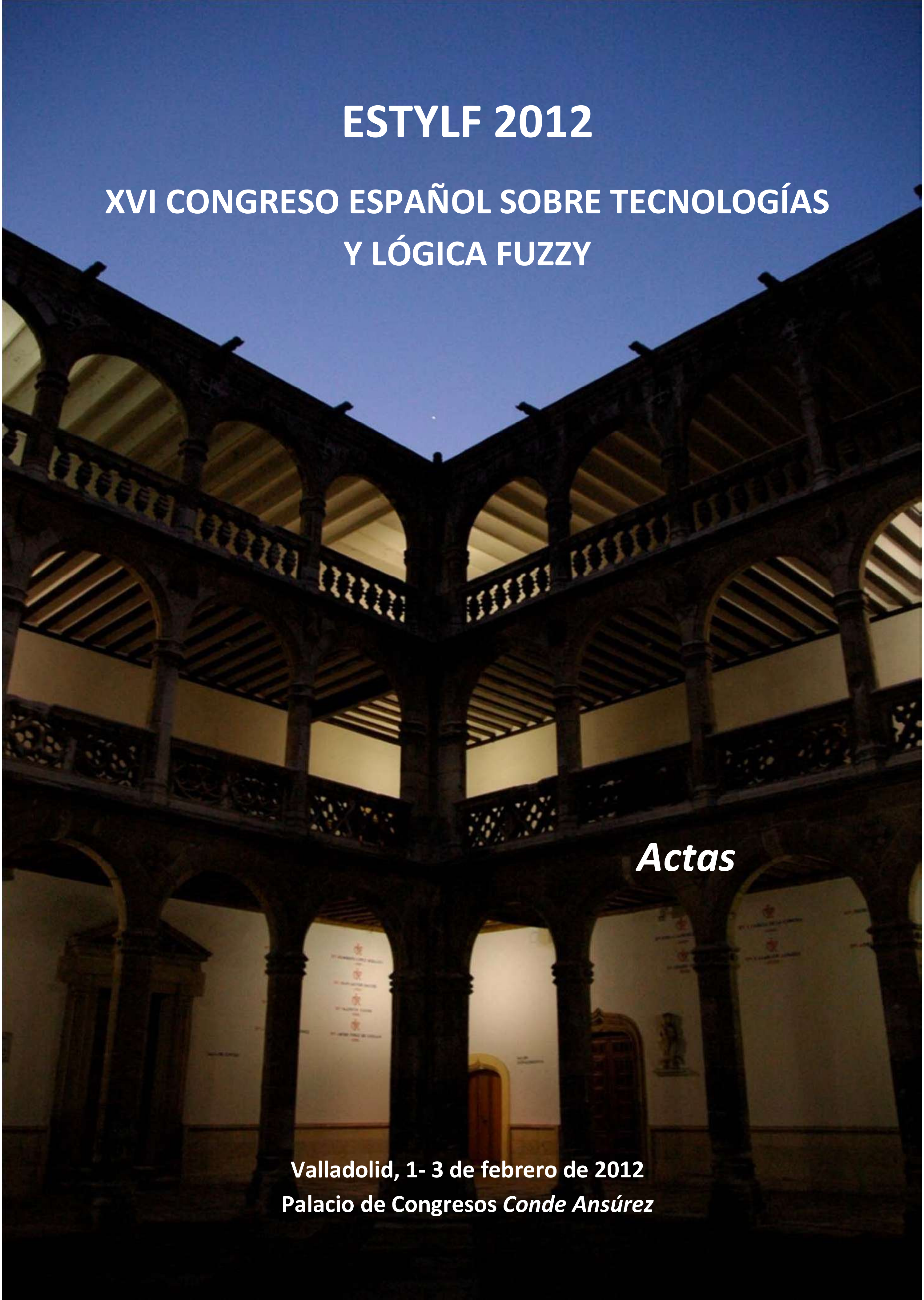

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A RECURSIVE ALGORITHM TO COMPUTE A BASIS OF A SIMILARITY

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Abstract

This paper studies some theory and methods to build a representation theorem basis of a similarity from the basis of its subsimilarities, providing an alternative recursive method to compute the basis of a similarity.

Keywords: T-indistinguishabilities representation theorem, similarity, basis of a similarity, dimension of a similarity, similarity representation theorem.

1 INTRODUCTION

Similarity fuzzy relations were introduced by Zadeh [1971] [10] to represent a degree of equality or closeness between the elements of a universe. They generalises the equivalence relations; in fact, the similarities are the only T-indistinguishabilities that satisfy that all their alpha cuts are crisp equivalence relations. Similarities are not only a powerful tool to represent equality information, but they are also useful to classify the universe into clusters with uncertainty.

The Valverde´s representation theorem of T-indistinguisabilities is one of the strongest theorems in fuzzy logic theory. It opened an interesting area of investigation on T-indistinguishabilities, the computation of its basis, specially for both the minimum t-norm and Archimedean t-norms, and the study of special T-indistinguishabilities such as the one dimensional ones.

On the other hand, the computational investigation on the computation of T-transitive closures, or the search of the structure of a similarity, specially its decomposition into subsimilarities, were not related in previous studies with the computation of the basis of a similarity.

Joan Jacas [4] studied two algorithms to compute basis of a similarity in 1990, but did not consider the decomposition of similarities to do it. The aim of this paper is to introduce a new approach to compute them, providing useful theory and methods for both better understanding the concept of structure of a similarity and the Valverde's representation theorem for similarities and the computation of their basis using a new decomposition approach.

2 PRELIMINARIES

Let $X = \{x_1, ..., x_n\}$ be a finite universe.

Let T be a t-norm. A T-indistinguishability operator E on X is a fuzzy relation $E: X \times X \to [0, 1]$, satisfying for all x, y, z in X:

1. $E(x, x) = 1$ (Reflexivity)
2. $E(x, y) = E(y, x)$ (Symmetry)
3. $T(E(x, y), E(y, z)) \leq E(x, z)$ (T-transitivity)

Definition 2.2. A **similarity** [Zadeh, 1971] [10] is a reflexive, symmetric and min-transitive fuzzy relation, it is, a similarity is a Min-indistinguishability operator.

Notation 2.1.

We can denote $x_{ij} = E(x_i, x_j)$.

Lemma 2.1. Let π be a permutation on X. If E is a similarity on X, then the fuzzy relation $P_\pi(E)$ is also a fuzzy similarity.

Proof. It is obvious. $P_\pi(E)$ is reflexive and symmetric. If $x_{ij} \geq \mathrm{Min}\{x_{ik}, x_{kj}\}$ for all i, j, k then $x_{rs} = x_{\pi(i)\ \pi(j)} \geq \mathrm{Min}\{x_{\pi(i)\ k}, x_{k\pi(j)}\} = T\{x_{rk}, x_{ks}\}$ for all $1 \leq r, s, k \leq n$.

2.1. CONSTRUCTION OF A FUZZY SIMILARITY FROM SUBSIMILARITIES

Let C and D be two similarities on two disjoint sets X_1 and X_2. A similarity relation E(F; C, D) on $X_1 \cup X_2$ can be built with the following shape:

$$E(F; C, D) = \begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$$

A method for giving the bridging values f_{ij} in F, (when $j \leq card(X_1) < i$) is the assignation of a unique value f, in all the $card(X_1) \times card(X_2)$ values in F. This value must be chosen in an interval [0, a] where a = min{min(C), min(D)}. The values in F^T are the symmetric values f of the computed F.

So the computed values in F are equal and satisfy that $f \leq \min\{\min(C), \min(D)\}$.

Lemma 2.1.1. [5] Lee [2001]

If C and D are fuzzy similarities, then E(f; C, D) is also a fuzzy similarity, $\forall f \in [0, \min(\min(C), \min(D))]$.

Lemma 2.1.2. [5] Lee [2001]

If $S_{n\times n}$ is a fuzzy similarity, then there exists a decomposition such that $S_{n\times n} = P_\pi(E(f; C_{n1\times n1}, D_{n2\times n2}))$.

2.2. THE REPRESENTATION THEOREM OF T-INDISTINGUISHABILITY OPERATORS

The representation theorem allow us to generate a *T*-indistinguishability operator on a set X from a family of subsets on X, and reciprocally states that every *T*-indistinguishability can be obtained in this form.

Definition 2.2.1 The residuation $\vec{T}$ or quasi inverse of a t-norm *T* is the map $\vec{T}$: $[0,1] \times [0,1] \to [0,1]$ defined for all x, y in [0, 1] by

$$\vec{T}\,(x \mid y) = \sup\{\propto \in [0,1] \mid T(x,\alpha) \le y\}.$$

The residuation $\vec{T}$ of a t-norm *T* is a *T*-preorder (reflexive and *T*-transitive) on [0, 1], and then it is a useful operator to generate implication relations from fuzzy sets on X.

Note that $\overrightarrow{Min}\,(x \mid y) = \begin{cases} 1 & if\ x \le y \\ y & if\ x > y \end{cases}$.

Definition 2.2.2 The biresiduation $\overleftrightarrow{T}$ (or also E_T) of a t-norm *T* is the map $\overleftrightarrow{T}$: $[0,1] \times [0,1] \to [0,1]$ defined for all x, y in [0, 1] by

$\overleftrightarrow{T}\,(x,\ y) = E_T(x,y) = Min\,(\vec{T}\,(x \mid y),\ \vec{T}\,(y \mid x))$.

Note that $\overleftrightarrow{Min}\,(x,\ y) = \overrightarrow{Min}(\text{Max}(x,\ y),\ \text{Min}(x,y))$

$= \begin{cases} 1 & if\ x = y \\ Min(x,y) & if\ x \ne y \end{cases}$.

The biresiduation $\overleftrightarrow{T}$ of a t-norm *T* is a *T*-indistinguishability operator on [0, 1], and then it is a useful operator to generate *T*-indistinguishability relations on X from two fuzzy sets on X.

Lemma 2.2.1.

Let μ be a fuzzy set on X, and *T* a continuous t-norm. The fuzzy relation E_μ on X defined for all x, y in X by $E_\mu(x,y) = E_T(\mu(x),\mu(y))$ is a T-indistinguishability operator.

Note that E_μ is a one-dimensional *T*-indistinguishability generated by a basis of one fuzzy set μ.

Lemma 2.2.2.

Let $(E_i)_{i\in I}$ be a family of *T*-indistinguishability operators on a set X. The relation E on X defined for all x, y in X by

$$E(x,y) = \inf_{i\in I} E_{\mu i}(x,y)$$

is a T-indistinguishability operator.

The next theorem is crucial to understand the structure of a T-indistinguishability operator. It allows us to generate *T*-indistinguishabilities from a family of fuzzy sets, and reciprocally that any *T*-indistinguishability can be generated from a family of fuzzy sets.

Representation theorem 2.2.1. [9]

Let R be a fuzzy relation on X and T a continuous t-norm. R is a T-indistinguishability operator if and only if there exists a family $(\mu_i)_{i\in I}$ of fuzzy sets on X such that for all x, y in X

$$R(x,y) = \inf_{i\in I} E_{\mu i}(x,y).$$

Definition 2.2.1. Dimension and basis of a T-indistinguishability operator

The dimension of a *T*- indistinguishability operator E is the minimal of the cardinalities *d* of the generating families of E, in the sense of the representation theorem.

That minimal basis of generating fuzzy sets is called a basis of the *T*-indistinguishability operator.

3 DECOMPOSITION OF SIMILARITIES AND REPRESENTATION THEOREM

This chapter gives some theoretical results toward a method to compute a basis of a similarity from the bases of its subsimilarities.

Lemma 3.1. Let S be the following similarity on $X_1 \cup X_2$,

S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$

Then dim(S) ≥ dim(C).

Lemma 3.2.

Let $(\mu_i)_{i\in I}$ be a representation theorem basis of a similarity C of dimension r on a finite set X_1. Let $(\gamma_i)_{i\in J}$ be a representation theorem basis of a similarity D of dimension s on a finite set X_2.

Let S be the following similarity on $X_1 \cup X_2$, S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$ where F = $\begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}$ is a $card(X_1) \times card(X_2)$ matrix.

Suppose that r > s and *f* < min(C) and *f* ≤ min(D) then dim(S) ≤ dim(C) and a generator set of S is $\left\{\begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_s) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (\mu_{s+1}) \\ F' \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ F' \end{pmatrix}\right\}$ where F' = $\begin{pmatrix} f \\ \cdot \\ \cdot \\ f \end{pmatrix}$ is a $card(X_2) \times 1$ matrix.

Proposition 3.1.

Let $(\mu_i)_{i\in I}$ be a representation theorem basis of a similarity C of dimension r on a finite set X_1. Let $(\gamma_i)_{i\in J}$ be a representation theorem basis of a similarity D of dimension s on a finite set X_2.

Let S be the following similarity on $X_1 \cup X_2$, S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$ where F = $\begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}$ is a $card(X_1) \times card(X_2)$ matrix.

Suppose that r > s and f < min(C) and $f \le$ min(D) then

1) dim(S) = dim(C) = r
2) a basis of S is $\left\{\begin{pmatrix}(\mu_1)\\(\gamma_1)\end{pmatrix}, \ldots, \begin{pmatrix}(\mu_s)\\(\gamma_s)\end{pmatrix}, \begin{pmatrix}(\mu_{s+1})\\F'\end{pmatrix}, \ldots, \begin{pmatrix}(\mu_r)\\F'\end{pmatrix}\right\}$

where F' = $\begin{pmatrix} f \\ \cdot \\ \cdot \\ f \end{pmatrix}$ is a $card(X_2) \times 1$ matrix.

Proof:

By Lemma 3.1 dim(S) $\ge$ dim(C).

By Lemma 3.2 dim(S) $\le$ dim(C). Also dim(S) = dim(C) = r and

$\left\{\begin{pmatrix}(\mu_1)\\(\gamma_1)\end{pmatrix}, \ldots, \begin{pmatrix}(\mu_s)\\(\gamma_s)\end{pmatrix}, \begin{pmatrix}(\mu_{s+1})\\F'\end{pmatrix}, \ldots, \begin{pmatrix}(\mu_r)\\F'\end{pmatrix}\right\}$ is a basis of S(F; C, D). □

Corollary 3.1.

Let $(\mu_i)_{i\in I}$ be a representations theorem basis of a similarity C on a finite set X = $\{x_1, \ldots, x_n\}$.

Let S be the similarity on X $\cup \{x_{n+1}\}$ such that S(F; C, 1) = $\begin{pmatrix} C & F'^T \\ F' & 1 \end{pmatrix}$ where F' = $(f \quad \ldots \quad f)$ is a $1 \times card(X)$ matrix.

If f < min(C), then a basis of S is

$\{\mu_1, \ldots, \mu_{\dim(C)}\} \times \{f\} = \left\{\begin{pmatrix}\begin{pmatrix}\mu_1(x_1)\\ \vdots \\ \mu_1(x_n)\end{pmatrix}\\ f\end{pmatrix}, \ldots, \begin{pmatrix}\begin{pmatrix}\mu_{\dim(C)}(x_1)\\ \vdots \\ \mu_{\dim(C)}(x_n)\end{pmatrix}\\ f\end{pmatrix}\right\}$

Example 3.1.

Let S be the similarity $\begin{pmatrix} 1 & a & b & c \\ a & 1 & b & c \\ b & b & 1 & c \\ c & c & c & 1 \end{pmatrix}$, already ordered with c < b < a.

S= $\begin{pmatrix} \begin{pmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{pmatrix} & \begin{matrix} c \\ c \\ c \end{matrix} \\ \begin{matrix} c & c & c \end{matrix} & 1 \end{pmatrix}$,

$and\ a\ base\ of$ $\begin{pmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{pmatrix}$ is $\left\{\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}\right\}$, then by the theorem 3.1 a basis of S is

$\left\{\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}\right\} \times \{c\} = \left\{\begin{pmatrix} 1 \\ a \\ b \\ c \end{pmatrix}\right\}$.

Note that S is one dimensional.

Example 3.2.

Let S be the similarity $\begin{pmatrix} 1 & a & a & b \\ a & 1 & a & b \\ a & a & 1 & b \\ b & b & b & 1 \end{pmatrix}$, already ordered with b < a.

S= $\begin{pmatrix} \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix} & \begin{matrix} b \\ b \\ b \end{matrix} \\ \begin{matrix} b & b & b \end{matrix} & 1 \end{pmatrix}$,

A basis of $\begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$ is $\left\{\begin{pmatrix} 1 \\ a \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}\right\}$

Then, as b < min$\begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}$, by theorem 3.1, a basis of S is $\left\{\begin{pmatrix} 1 \\ a \\ a \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}\right\} \times \{b\} = \left\{\begin{pmatrix}\begin{pmatrix} 1 \\ a \\ a \end{pmatrix}\\ b\end{pmatrix}, \begin{pmatrix}\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix}\\ b\end{pmatrix}\right\}$

Lemma 3.2.

Let $(\mu_i)_{i\in I}$ be a representation theorem basis of a similarity C of dimension r on a finite set X_1. Let $(\gamma_i)_{i\in J}$ be a representation theorem basis of a similarity D of dimension s on a finite set X_2.

Let S be the following similarity on $X_1 \cup X_2$, S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$, where F= $\begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}$ is a $card(X_1) \times card(X_2)$ matrix

Suppose that r > s, f = min(C) and $f \le$ min(D), then:

1) dim(S) > dim(C) = r.

Lemma 3.3.

Let $(\mu_i)_{i\in I}$ be a representation theorem basis of a similarity C of dimension r on a finite set X_1 Let $(\gamma_i)_{i\in J}$ be a representation theorem basis of a similarity D of dimension s on a finite set X_2.

Let S be the following similarity on $X_1 \cup X_2$, S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$,

where F = $\begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}$ is a $card(X_1) \times card(X_2)$ matrix

Suppose that r > s, f = min(C) and f ≤ min(D), then:

1) dim(S) ≤ dim(C) +1 = r + 1
2) A generator set of S is A basis of S is $\left\{ \begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_s) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (\mu_{s+1}) \\ F' \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ F' \end{pmatrix}, \begin{pmatrix} (1) \\ F' \end{pmatrix} \right\}$,

where $(1) = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$ is a $card(X_1) \times 1$ matrix and F' = $\begin{pmatrix} f \\ \cdot \\ \cdot \\ f \end{pmatrix}$ is a $card(X_2) \times 1$ matrix.

Proof:

Now $\left\{ \begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_s) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (\mu_{s+1}) \\ F' \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ F' \end{pmatrix}, \begin{pmatrix} (1) \\ F' \end{pmatrix} \right\}$ is a generator set of S, because

$inf \left\{ \inf_{1 \le i \le s} E_{\begin{pmatrix} (\mu_i) \\ (\gamma_i) \end{pmatrix}}, \inf_{r+1 \le j \le r} E_{\begin{pmatrix} (\mu_j) \\ F \end{pmatrix}} \right\} = \begin{pmatrix} C & F* \\ F* & D \end{pmatrix}$ = S(F; C, D).

Also $\begin{pmatrix} (1) \\ F' \end{pmatrix} o_{\overrightarrow{Min}} ((1), \ F') = \begin{pmatrix} (1) & F^T \\ F & (1) \end{pmatrix}$, so the infimum of all the similarities is $\begin{pmatrix} C & F^T \\ F & D \end{pmatrix}$ = S(F; C, D).□

Proposition 3.2.

Let $(\mu_i)_{i \in I}$ be a representation theorem basis of a similarity C of dimension r on a finite set X_1. Let $(\gamma_i)_{i \in J}$ be a representation theorem basis of a similarity D of dimension s on a finite set X_2.

Let S be the following similarity on $X_1 \cup X_2$, S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$, where F= $\begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}$ is a $card(X_1) \times card(X_2)$ matrix

Suppose that r > s, f = min(C) and f ≤ min(D), then:

1) dim(S) = dim(C) +1 = r + 1
2) A basis of S is:

$\left\{ \begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_s) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (\mu_{s+1}) \\ F' \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ F' \end{pmatrix}, \begin{pmatrix} (1) \\ F' \end{pmatrix} \right\}$

where $(1) = \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$ is a $card(X_1) \times 1$ matrix and F' = $\begin{pmatrix} f \\ \cdot \\ \cdot \\ f \end{pmatrix}$ is a $card(X_2) \times 1$ matrix.

Proof:

By lemma 3.2 dim(S) > dim(C) = r, and by lemma 3.3 dim(S) ≤ dim(C) +1 = r + 1, then dim(S) = dim(C) +1 = r + 1, and a basis of S is $\left\{ \begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_s) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (\mu_{s+1}) \\ F' \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ F' \end{pmatrix}, \begin{pmatrix} (1) \\ F' \end{pmatrix} \right\}$,

Example 3.5

Let S be the similarity $\begin{pmatrix} 1 & a & b & b \\ a & 1 & b & b \\ b & b & 1 & b \\ b & b & b & 1 \end{pmatrix}$, already ordered with b < a.

S = $\begin{pmatrix} \begin{pmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{pmatrix} & \begin{matrix} b \\ b \\ b \end{matrix} \\ \begin{matrix} b & b & b \end{matrix} & 1 \end{pmatrix}$, and a basis of $\begin{pmatrix} 1 & a & b \\ a & 1 & b \\ b & b & 1 \end{pmatrix}$, is $\left\{ \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \right\}$, then by the propositions 3.1 and 3.2 a generator set of S is = $\left\{ \begin{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \\ b \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ b \end{pmatrix} \right\}$.

Proposition 3.3.

Let $(\mu_i)_{i \in I}$ be a representation theorem basis of a similarity C of dimension r on a finite set X_1. Let $(\gamma_i)_{i \in J}$ be a representation theorem basis of a similarity D of dimension s on a finite set X_2.

Let S be the following similarity on $X_1 \cup X_2$, S(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$,

where F = $\begin{pmatrix} f & \cdots & f \\ \vdots & \ddots & \vdots \\ f & \cdots & f \end{pmatrix}$ is a $card(X_1) \times card(X_2)$ matrix.

Suppose that $r = s$ (dim(C) = dim(D) and f < min(C, D), then a generator set of S is:

$$\left\{ \begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (1) \\ F' \end{pmatrix} \right\}$$

and dim(S) ≤ r + 1.

Proof:

Let $\{\mu_1, \dots, \mu_r\}$ be a representation theorem basis of a similarity C, so

C = $\inf_{i \in I} E_{\mu i}$ where I = {1, ..., r}.

In the same way, let $\{\gamma_1, ..., \gamma_s\}$ be a representation theorem basis of a similarity D, so

D = $\inf_{j\in J} E_{\gamma j}$ where J = {1, ..., r}

Now, for all $1 \leq i \leq r$,

$$\begin{pmatrix}(\mu_i)\\(\gamma_i)\end{pmatrix} o_{\overrightarrow{Min}}((\mu_i) \quad (\gamma_i)) = \begin{pmatrix}(E_{\mu i}) & (x_{nm})\\(x_{mn}) & (E_{\gamma i})\end{pmatrix}$$

Also $\begin{pmatrix}(1)\\F'\end{pmatrix} o_{\overrightarrow{Min}}((1) \quad F') = \begin{pmatrix}(1) & F^T\\F & (1)^T\end{pmatrix}$,

So the infimum of all the similarities is $\begin{pmatrix}C & F^T\\F & D\end{pmatrix} = S(F; C, D)$, and then $\left\{\begin{pmatrix}(\mu_1)\\(\gamma_1)\end{pmatrix}, ..., \begin{pmatrix}(\mu_r)\\(\gamma_s)\end{pmatrix}, \begin{pmatrix}(1)\\F'\end{pmatrix}\right\}$ is a generator set of S. □

Example 3.4.

Let S be the similarity $\begin{pmatrix}1 & a & b & b\\a & 1 & b & b\\b & b & 1 & a\\b & b & a & 1\end{pmatrix}$, already ordered, with b < a.

$$S = \begin{pmatrix}\begin{pmatrix}1 & a\\a & 1\end{pmatrix} & \begin{matrix}b & b\\b & b\end{matrix}\\\begin{matrix}b & b\\b & b\end{matrix} & \begin{pmatrix}1 & a\\a & 1\end{pmatrix}\end{pmatrix}$$

A basis of $\begin{pmatrix}1 & a\\a & 1\end{pmatrix}$, is $\left\{\begin{pmatrix}1\\a\end{pmatrix}\right\}$, then by the proposition 3.3 a generator set (and a basis in this example) of S is $\left\{\begin{pmatrix}(\mu_1)\\(\gamma_1)\end{pmatrix}, ..., \begin{pmatrix}(\mu_r)\\(\gamma_s)\end{pmatrix}, \begin{pmatrix}(1)\\F'\end{pmatrix}\right\} = \left\{\begin{pmatrix}1\\a\\1\\a\end{pmatrix}, \begin{pmatrix}1\\1\\b\\b\end{pmatrix}\right\}$.

Note that there exists other bases of S, for example $\left\{\begin{pmatrix}1\\a\\b\\b\end{pmatrix}, \begin{pmatrix}b\\b\\1\\a\end{pmatrix}\right\}$.

Let S be the similarity $\begin{pmatrix}1 & a & b & b\\a & 1 & b & b\\b & b & 1 & a\\b & b & a & 1\end{pmatrix}$, already ordered, with b < a.

$$S= \begin{pmatrix}\begin{pmatrix}1 & a\\a & 1\end{pmatrix} & \begin{matrix}b & b\\b & b\end{matrix}\\\begin{matrix}b & b\\b & b\end{matrix} & \begin{pmatrix}1 & a\\a & 1\end{pmatrix}\end{pmatrix}$$

$A\ base\ of\ \begin{pmatrix}1 & a\\a & 1\end{pmatrix}\ is\ \left\{\begin{pmatrix}1\\a\end{pmatrix}\right\}$,

$Then$ by proposition 3.3 a basis of S is

$$=\left\{\begin{pmatrix}1\\a\\b\\b\end{pmatrix} \begin{pmatrix}b\\b\\1\\a\end{pmatrix}\right\}.$$

4 GENERATIING A DECOMPOSITION OF A GIVEN SIMILARITY

Input: A similarity S on X.

Output:

- Two similarities S_1, and S_2 on X_1 and X_2 such that X = $X_1 \cup X_2$
- A bridging value f such that f < min(S_1) and f ≤ min(S_2)

Algorithm 5.1:

1. Sort the universe X having descending columns under the diagonal using algorithm 4.1
2. Compute the frequency k of the lowest value v in the first column
3. If k ≥ card(X) -1 then STOP. S_1 = S.
4. Let X_1 be {x_1, ..., x_{k+1}} and let X_2 be {x_{k+2}, ..., $x_{card(X)}$}
5. Let S_1 = S| X_1, S_2 = S| X, and f = v.

5 A RECURSIVE ALGORITHM TO COMPUTE A GENERATING SET OF A SIMILARITY

Input: a similarity S on a universe X.

Output: A generating set of S, and so, an upper bound of the dimension of S.

A recursive algoritm:

1) If card(X) ≤ 2 then STOP. dim(S) = 1 and a basis of S is the first column.
2) Descompose S into S_1, and S_2 using algorithm 5.1. Note that f < min(S_1)
3) IF card(X_2) = 1 then use corollary 3.1:
   a. Recursively compute a generating set of S_1, $(\mu_i)_{i\in I}$
   b. A generating set of S is $\{(\mu_i)_{i\in I}\} \times \{f\}$
   c. STOP
4) Recursively compute a generating set of S_1, $(\mu_i)_{i\in I}$ and S_2, $(\gamma_i)_{i\in J}$
5) IF dim(S_1) > dim(S_2) -or vice versa- then use proposition 3.1
   a. A generating set of S is $\left\{\begin{matrix}\begin{pmatrix}(\mu_1)\\(\gamma_1)\end{pmatrix} \cdots \begin{pmatrix}(\mu_s)\\(\gamma_s)\end{pmatrix} \begin{pmatrix}(\mu_{s+1})\\F\end{pmatrix} \cdots\\ \begin{pmatrix}(\mu_r)\\F\end{pmatrix}\end{matrix}\right\}$
   b. STOP
6) IF dim(S_1) = dim(S_2) then use proposition 3.3
   a. A generating set of S is $\left\{\begin{pmatrix}(\mu_1)\\(\gamma_1)\end{pmatrix} \cdots \begin{pmatrix}(\mu_r)\\(\gamma_s)\end{pmatrix} \begin{pmatrix}(1)\\F\end{pmatrix}\right\}$
   b. STOP

Example 6.1.

Let S be the sorted similarity with descending columns under the diagonal given by the following matrix:

$$S = \begin{pmatrix} 1 & 0,9 & 0,4 & 0,4 & 0,3 \\ 0,9 & 1 & 0,4 & 0,4 & 0,3 \\ 0,4 & 0,4 & 1 & 0,7 & 0,3 \\ 0,4 & 0,4 & 0,7 & 1 & 0,3 \\ 0,3 & 0,3 & 0,3 & 0,3 & 1 \end{pmatrix}$$

It will be computed the algorithm 6.1 step by step.

A decomposition of S is given by algorithm 5.1 is

$$S = \begin{pmatrix} \begin{pmatrix} 1 & 0,9 & 0,4 & 0,4 \\ 0,9 & 1 & 0,4 & 0,4 \\ 0,4 & 0,4 & 1 & 0,7 \\ 0,4 & 0,4 & 0,7 & 1 \end{pmatrix} & \begin{matrix} 0,3 \\ 0,3 \\ 0,3 \\ 0,3 \end{matrix} \\ \begin{matrix} 0,3 & 0,3 & 0,3 & 0,3 \end{matrix} & (1) \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0,9 & 0,4 & 0,4 \\ 0,9 & 1 & 0,4 & 0,4 \\ 0,4 & 0,4 & 1 & 0,7 \\ 0,4 & 0,4 & 0,7 & 1 \end{pmatrix},$$

Now that S is ordered and decomposed, we can build its basis as follows:

A basis of $\begin{pmatrix} 1 & 0,9 \\ 0,9 & 1 \end{pmatrix}$ is $\left\{\begin{pmatrix} 1 \\ 0,9 \end{pmatrix}\right\}$. A basis of $\begin{pmatrix} 1 & 0,7 \\ 0,7 & 1 \end{pmatrix}$ is $\left\{\begin{pmatrix} 1 \\ 0,7 \end{pmatrix}\right\}$, and dimC = 2 = dimD, f = 0,4 < min(C, D). Then by proposition 3.3, a generator set of S_1 =

$$\begin{pmatrix} \begin{pmatrix} 1 & 0,9 \\ 0,9 & 1 \end{pmatrix} & \begin{matrix} 0,4 & 0,4 \\ 0,4 & 0,4 \end{matrix} \\ \begin{matrix} 0,4 & 0,4 \\ 0,4 & 0,4 \end{matrix} & \begin{pmatrix} 1 & 0,7 \\ 0,7 & 1 \end{pmatrix} \end{pmatrix}$$

is $\left\{\begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (1) \\ F' \end{pmatrix}\right\} = \left\{\begin{pmatrix} 1 \\ 0.9 \\ 1 \\ 0.7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0,4 \\ 0,4 \end{pmatrix}\right\}$

And then, by proposition 3.1, a generator set of S = $\begin{pmatrix} S_1 & 0,3 \\ 0,3 & (1) \end{pmatrix}$ is

$$\left\{\begin{pmatrix} (\mu_1) \\ (\gamma_1) \end{pmatrix}, \dots, \begin{pmatrix} (\mu_s) \\ (\gamma_s) \end{pmatrix}, \begin{pmatrix} (\mu_{s+1}) \\ F' \end{pmatrix}, \dots, \begin{pmatrix} (\mu_r) \\ F' \end{pmatrix}\right\} =$$

$$\left\{\begin{pmatrix} 1 \\ 0,9 \\ 1 \\ 0,7 \\ 0,3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0,4 \\ 0,4 \\ 0,3 \end{pmatrix}\right\}$$

6 CONCLUDING REMARKS

This paper´s main contribution is a method to compute a representation theorem basis of a similarity from the bases of its subsimilarities.

These results can be used to propose an alternative algorithm to build a basis of similarities.

The bases of all structures of similarities with dimension four are computed in the examples using the new construction method.

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8 REFERENCES

[1]. J. L. Castro, F. Klawonn, F.: Similarity in Fuzzy Reasoning. *Mathware & Soft Computing* 2, 197-228, 1996

[2]. J. C. Fodor, M. Roubens: Structure of transitive valued binary relations. *Math. Soc. Sci. 30*,71-94, 1995.

[3]. J. Jacas: Similarity relations. The calculation of minimal generating families. *Fuzzy Sets and Systems* 35, 151-162, 1990.

[4]. J. Jacas, R. Recasens: J. One dimensional indistinguishability operators. *Fuzzy Sets and Systems* 109, 447-451, 2000

[5]. H. S. Lee: An optimal algorithm to compute the max-mintransitive closure of a fuzzy similarity. *Fuzzy Sets and Systems* 123, 129-136, 2001.

[6]. J. Recasens: *Indistinguishability operators: Modelling Fuzzy Equallities and Fuzzy Equivalence relations.* Studies in Fuzziness and Soft Computing. Springer. 2010.

[7]. E. Trillas: Assaig sobre les relacions d'indistingibilitat. *Proc. Primer Congrés Català de Lògica Matemàtica*, Barcelona, 51-59, 1982

[8]. E. Trillas, L. Valverde: An inquiry into indistinguishability operators. *Aspects of vagueness,* 231-256, Theory Decis. Lib., 39, Reidel, Dordrecht, 1984.

[9]. L. Valverde: On the structure of F-indistinguishability Operators. *Fuzzy Sets and Systems* 17, 313-328, 1985.

[10]. L. A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sci*. 3, 177–200, 1971.